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Asymptotic Integration of the Solutions of Unsteady Boundary Layer and Heat Transfer Equations due to a Stretching Sheet

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***Abstract*— This paper deals with the asymptotic behaviours of the solutions of the unsteady boundary layer heat transfer equations governing the distributions of the stretching velocity and surface temperature or surface heat flux. The method to be used here is the asymptotic integration of second order linear differential equations.**

***Index Terms*— Asymptotic behaviour; boundary layer equations; heat transfer. 2010 Mathematics Subject Classification: 80-XX, 80Axx, 35Q79, 76F40, 37K40, 58K55, 35B40 (2010 MSC)**

I. INTRODUCTION

The flow and heat transfer of viscous and incompressible fluid induced by a continuously moving or stretching surface in a quiescent fluid is relevant to many manufacturing processes. A number of technical processes concerning polymers involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. Further glass blowing, continuous casting of metals and spinning of fibers involve the flow due to a stretching surface. In these cases, the property of the final product depend to a great extent on the rate of cooling which is governed by the structure of the boundary layer near the moving strip. Crane [1] seemed to initiate the study of the boundary layer flow due to a stretching surface in an otherwise ambient fluid. Carrangher and Crane [2] investigated the heat transfer in the flow over a stretching surface in the case when the temperature difference between the surface and the ambient fluid is proportional to a power of distance from the fixed point. Similar flow and heat transfer problems were studied by researchers like Dutta et al. [3], Grubka and Bobba [4], Elbashbeshy [5], Lin and Chen [6], Gupta and Gupta [7], Chen and Char [8], Magyari and Keller [9-8], Liao and Pop [11], Na and Pop [12], Wang et al. [13], Nazar et al. [14], Elbashbeshy and Bazid [15].

The objective of the present paper is to study the asymptotic behaviours of the solutions of the similarity boundary layer flow and heat transfer equations over a stretching sheet in a viscous and incompressible fluid which is at rest under the similarity conditions considered by Elbashbesy and Bazid [15]. The asymptotic behaviours have been studied by using some topological arguments related to the existence and uniqueness of the solutions, and asymptotic integrations of second order linear differential equations. The results pertaining to the existence and uniqueness of the solutions have been expressed in terms of theorems.

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The study of the existence, uniqueness and asymptotic behaviours of the solutions of the equations governing the flow problems of physical significance in boundary layer theory is an interesting aspect of discussion in fluid mechanics. The existence and uniqueness of the solutions of the Falknar-Skan [16] equations and their types, which govern various flow problems, were studied by Hartree [17], Weyl [18], Stewartson [19], Rosenhead [20], Hasting [21], Troy [22], Gabutti [23], Singh and Chandarki [24], etc. The study of the asymptotic behaviour was initiated by Hartman [28], and later extended by Singh and Kumar [26], Singh [27] and by various other authors.

II. Mathematical Formulation

Let us consider the unsteady flow and heat transfer of a viscous and incompressible fluid past a semi-infinite stretching sheet in the region y>0 , as shown in Fig 1.

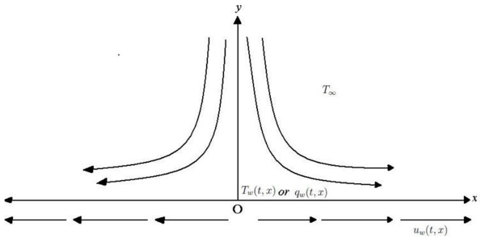

Figure 1. Physical model and co-ordinate system

Keeping the origin fixed, two equal and opposite forces are suddenly applied along the x- axis resulting in the stretching of the sheet and hence flow is generated. At the same time the wall temperature $T_w(t,x)$ of the sheet is suddenly raised from T_∞ to $T_w(t,x)(>T_\infty)$ or suddenly imposed a heat flux $q_w(t,x)$ at the wall. Under these assumptions, the basic unsteady boundary layer equations governing the flow and heat transfer due to the stretching sheet are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \vartheta\frac{\partial^2 u}{\partial y^2} \tag{2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{2.3}$$

subject to the boundary conditions

$$\begin{cases} t<0: & u=v=0 \qquad T=T_\infty \quad \text{any} \quad x,y \\ t>0: & u=u_w(t,x), \qquad v=0, \\ T=T_w(t,x), & \frac{\partial T}{\partial y}=-\frac{q_w(t,x)}{\kappa}, \\ u\to 0, & T\to T_\infty \quad \text{as} \qquad y\to\infty \end{cases} \tag{2.4}$$

where t is time; u,v are the velocity components along x −axis and y −axis, respectively; T is temperature; α is the thermal diffusivity; ϑ is the kinematic viscosity and κ is thermal conductivity.
Now we assume that the velocity of the sheet $u_w(t,x)$, the sheet temperature $T_w(t,x)$, and the heat flux $q_w\,(t,x)$ have the following form:

$$\begin{cases} u_w(t,x) = cx(1-\gamma\,t)^{-1}, \; T_w(t,x) = T_\infty + \frac{c}{c\vartheta\,x^2}(1-\gamma\,t)^{-\frac{3}{2}} \\ q_w(t,x) = \frac{q_{w_0}}{2x^2}\left(\frac{c}{\vartheta}\right)^{\frac{3}{2}}(1-\gamma\,t)^{-2} \end{cases} \tag{2.5}$$

where c is the stretching rate and is a positive constant. Also γ is positive constant measuring the unsteadiness and q_{w_0} is the characteristic heat transfer quantity.

We now use the following new variables:

where ψ is the stream function which is defined in the usual way as $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$.

Substituting (2.6) in the equations (2.2) and (2.3), they reduce to the following ordinary differential equations:

$$f''' + \left(f - \frac{A\eta}{2}\right)f'' - (A + f')f' = 0 \tag{2.7}$$

$$\theta'' + P_r(f - A\eta)\theta' + P_r(2f' - 3A)\theta = 0 \tag{2.8}$$

subject to the boundary conditions (2.4), which become

$$\begin{cases} f(0) = 0,\ f'(0) = 1,\ \theta(0) = 1 \\ f'(\infty) = 0,\ \theta(\infty) = 0 \end{cases} \tag{2.9}$$

Let us put $f = \eta - F$ in (2.7), (2.8) and (2.9) to obtain

$$F''' + (\alpha_1 \eta - F)F'' + (\alpha_2 - F')(1 - F') = 0 \tag{2.10}$$

$$\theta'' + P_r(\alpha_3 \eta - F)\theta' + P_r(\alpha_4 - 2F')\theta = 0 \tag{2.11}$$

under the conditions

$$F(0) = F'(0) = 0,\quad F'(\infty) = 1 \tag{2.12}$$

$$\theta(0) = 1,\ \theta'(\infty) = 0 \tag{2.13}$$

where $\alpha_1 = 1 - \frac{A}{2}$, $\alpha_2 = 1 + A$, $\alpha_3 = 1 - A$, $\alpha_4 = 2 - 3A$.

Here P_r is the Prandtl number, $A = \frac{\gamma}{c}$ is a non-dimensional constant measuring the flow and heat transfer unsteadiness, and primes denote derivatives with respect to the similarity variable η.
The equations (2.10), (2.11) together with the conditions (2.12), (2.13) govern the required problem.

III. Asymptotic Behaviour

For the study of asymptotic behaviour of solutions of (2.10), (2.12) the following topological arguments are required:

Lemma 3.1. Let g, ξ be d −dimensional vectors and $\xi(\eta, g)$ continuous on an open (η, g) − set Ω such that the solutions of the initial value problems associated with

$$g' = f(\eta, g) \tag{3.1}$$

are unique. Let Ω_0 be an open subset of Ω with the properties that all egress points from Ω_0 are strict egress points and that the set Ω_e of egress points is not connected. Let Ω_i denote the set of ingress points of Ω_0 and S a connected subset of $\Omega_0 \cup \Omega_e \cup \Omega_i$ such that $S \cap (\Omega_0 \cup \Omega_i)$ contains two points (η_1, g_1), (η_2, g_2) for which the solutions $g_j(\eta)$ of (3.1) through (η_j, g_j) for $j = 1, 2$ leave Ω_0 with increasing η at points of different connected components of Ω_e. Then there exist at least one point $(\eta_0, g_0) \in S \cap (\Omega_0 \cup \Omega_i)$ such that the solution $g_0(\eta)$ of (3.1) determined by $g_0(\eta_0) = g_0$ remains in Ω_0 on its (open) right maximal interval of existence.

Proof. The proof of the lemma (3.1) is given (Hartman [17], p.520). For the definitions of egress, ingress and strict egress point let us see (Hartman [17], p.37).

Theorem 3.2. The boundary value problems (2.10), (2.12) has at least one solution $F(\eta)$ satisfying

$$0 < F' < 1,\ F'' > 0 \ on\ [0, \infty). \tag{3.2}$$

Proof. The proof of the theorem is based on Lemma (3.1). To this end, we rewrite the equation (2.10) in the system form. If we set $g_1 = F$, $g_2 = F'$, $g_3 = F'''$ then we have

$$\begin{cases} g_1{}' = g_2 \\ g_2' = g_3 \\ g_3' = g_1\, g_3 - [\alpha_2 - g_2] - \eta\, \alpha_1\, g_3 \end{cases} \quad (3.3)$$

$$\Omega = \{ (\eta, \quad g_1, g_2, g_3) : \eta, \quad g_1, \quad g_2, \quad g_3 \in \Re \}$$

and $\Omega_0 = \{ (\eta,\ g_1,\ g_2,\ g_3) : \eta, g_1 \in \Re,\ 0 < g_2 < 1, g_3 > 0 \}$

To determine the ingress and egress points, let us introduce the following boundary sets associated with Ω_0 :

$$\Omega_1 = \{ (\eta, g_1, g_2, g_3) : \eta , g_1 \in \Re,\ g_2 = 0,\ g_3 > 0 \}$$

$$\Omega_2 = \{ (\eta, g_1, g_2, g_3) : \eta, g_1 \in \Re, 0 < g_2 < 1, g_3 = 0 \}$$

$$\Omega_3 = \{ (\eta, g_1, g_2, g_3) : \eta , g_1 \in \Re, \quad g_2 = 1, g_3 > 0 \}$$

$$\Omega_4 = \{ (\eta, g_1, g_2, g_3) : \eta, g_1 \in \Re, g_2 = 1, g_3 = 0 \}$$

$$\Omega_5 = \{ (\eta, g_1, g_2, g_3) : \eta, g_1 \in \Re, g_2 = 0, g_3 = 0 \}$$

Here the set of ingress points is $\Omega_i = \Omega_1$. In fact in Ω_1 we have $g_2 = 0$ and $g_3 = g_2{}' > 0$.
The set of strict egress points is $\Omega_e = \Omega_2 \cup \Omega_3$. This follows from $g_2 = 1$, $g_3 = g_2{}' > 0$ for Ω_3 and from $g_3{}' = -(\alpha\_2 - g\_2)(1 - g_2) < 0$, for Ω_2 .
The set Ω_4 is composed of solution $g_1 = \eta + C$ (is constant); therefore, the points in Ω_4 are neither egress nor ingress points.
Thus Ω_e is not connected.
For points $(\eta, g_1, g_2, g_3) \in \Omega_5$
it holds that $g_2\,(\eta_0) = g_3(\eta_0) = 0$ and $g_3{}' = -\alpha_2$. These imply that $g_3(\eta)$, $g_2(\eta) < 0$ if $|\eta - \eta_0|$ is small enough. Thus the solution $(g_1,\ g_2,\ g_3)$ passing through $(\eta_0, g_0, 0, 0)$ is not in Ω_0.
Now, if k_1 is a fixed number satisfying $0 < k_1 < \infty$, let us set

$$S = \{(\eta, g_1, g_2, g_3) : \eta = 0, g_1 = 0, g_2 = 0, g_3 = k_1\}.$$

Clearly, S is connected subset of $\Omega_0 \cup \Omega_e \cup \Omega_i$.
The point $(0,0,0,k_2) \in S$, where k_2 is small and positive, is a strict ingress point of Ω_0 and the solution of (3.3) with $g_1(0) = g_2(0) = 0, g_3(0) = k_2$ leaves Ω_0 through the component Ω_2. Indeed, the solution of (3.3) with $g_1(0) = g_2(0) = 0, g_3(0) = k_2$ satisfies $g_3{}'(0) = -\alpha_2$ so that, by continuity of initial data, it follows $g_3(\eta) < 0, \eta > 0$ if k_2 is sufficiently small.
On the other hand, if $k_3 > 0$ is large enough, i.e. solution of (3.3) satisfying $g\_1(0) = g_2(0) = 0$, $g_3(0) = k_2$ leaves Ω_0 through a point in Ω_3 .
To verify this, let us note that $(\eta, g_1, g_2, g_3) \in \Omega$ implies that $g_3(\eta)$, $g_2(\eta) > 0$ and $0 \leq g_1(\eta) \leq \eta$ for some $\eta > 0$. Let us use it in the third equation of (3.3) and integrate to find $g_3(\eta) \geq k_3 - (\alpha_1 + \alpha_2)\eta$.
Hence if k_3 is sufficiently large and the solution of (3.3) through $(0,0,0,k_3)$ in Ω_0 on $[0,\eta_1)$ for some $\eta_1 > 0$, then $g_3(\eta)$ is greater than a given positive constant *on* $[0,\eta_1)$ and such a solution leaves Ω_0 through Ω_3. .
From the Lemma (3.1), it follows then that there exists a point $(0,0,0,\hat{k})$ in $S \cap (\Omega_0 \cup \Omega_1)$ such that the solution $(\widehat{g_1}, \widehat{g_2}, \widehat{g_3})$ of (3.3) with $\widehat{g_1}(0) = \widehat{g_2}(0) = 0$, $\widehat{g_3}(0) = \hat{k}$, remains in Ω_0 on its right maximal interval of existence. Because of the structure of Ω_0 , this is necessarily $[0,\infty)$.
Finally, we prove that

$$lim_{\eta\to\infty}\ \widehat{g_3}(\eta) = 0,\ \ \lim_{n\to\infty} \widehat{g_2}(\eta) = 1. \quad (3.4)$$

The first limit follows immediately by observing that if we suppose, for the purpose of obtaining a contradiction, that $\lim_{\eta\to\infty} \widehat{g_3}(\eta) = C \neq 0$, then we obtain $|\widehat{g_2}(\eta)| > 1$, which contradicts $(\eta , \widehat{p_1},\ \widehat{p_2}, \widehat{p_3}) \in \Omega_0$ for $\eta \in (0,\infty)$.

Analogously, let us suppose if possible, that $\lim_{\eta\to\infty} \widehat{g_2}(\eta) = \widehat{g_2}(\infty)$ where $0 < \widehat{g_2}(\infty) < 1$. The initial condition $\widehat{g_1}(\eta) = 0$ and structures of the sets Ω_2 , Ω_3, imply $\widehat{g_1}(\eta)\ and\ \widehat{g_1}(\eta)\widehat{g_2}(\eta) \geq 0$, $\eta \in [0,\infty)$.
The use of this and the first limit of (3.4) into the third equation of (3.3) gives

$$\lim_{\eta\to\infty} \widehat{g_3}(\eta) = \lim_{\eta\to\infty} [\widehat{g_1}(\eta)\widehat{g_2}(\eta) - \{\alpha_2 - \widehat{g_2}(\eta)\}\{1 - \widehat{g_2}(\eta)\} - \eta\, \alpha_1 \widehat{g_3}(\eta)]$$

$$\geq -\{\alpha_2 - \widehat{g_2}(\infty)\}\{1 - \widehat{g_2}(\infty)\} - \alpha_1(\infty) \neq 0$$

Two integrations lead to $\lim_{\eta\to\infty} \widehat{g_2}(\eta) = \infty$. This contradiction completes the proof of (3.4) showing that the inequalities in (3.2) are true. Thus the proof of Theorem 3.1 is completed. □

Theorem 3.3 There is a unique solution $F(\eta)$ of the boundary value problem (2.10), (2.12) such that $F'(\eta) > 0$.

Proof. Let us suppose, for the purpose of obtaining a contradiction, that there are two solutions $F_1(\eta)$ and $F_2(\eta)$ of (2.10), (2.12) such that $F_1(\eta) > 0$, $F_2(\eta) > 0$ on $(0,\infty)$. If $F_1(\eta) \neq F_2(\eta)$, we assume without loss of generality that $F_1'(\eta) > F_2'(\eta)$ on $(0,\eta_0)$ and $F_1'(\eta_0) = F'\_2(\eta_0)$ where $0 < \eta_0 < \infty$. Then (2.10) implies that $f_1(\eta) > f_2(\eta)$ on $[0,\eta_0)$.

If we now define $r(\eta)$ by $r(\eta) = F_1(\eta) - F_2(\eta)$, we see that

$$r(\eta),\ r'(\eta) > 0\ (0,\eta_0),\ r(0) = r(\eta_0) = 0 \tag{3.5}$$

Thus, $r'(\eta)$ has relative maximum occurring at some point $\eta_M \in (0,\eta_0)$ so that

$$r'(\eta_M) > 0,\ r''(\eta_M) = 0,\ r'''(\eta_M) \leq 0 \tag{3.6}$$

Moreover, since either F_1 or F_2 is the solutions of (2.10), (2.12) established by Theorem 3.1, hence from $F''(\eta) > 0$ on $(0,\infty)$ and the second inequality of (3.6) it follows that

$$F_1''(\eta_M) = F_2''(\eta_M) > 0,\ \eta_M \in (0,\eta_0) \tag{3.7}$$

Therefore, from (2.10), we obtain

$$r'''(\eta_M) = \{F_1(\eta_M) - \alpha_1\,\eta_M\}\, r''(\eta_M) + (\alpha_2 + 1)\, r'(\eta_M) + \{F_2''(\eta_M) - F_1'(\eta_M) - F_2'(\eta_M)\} r(\eta_M\ ) \qquad 3.8)$$

By using (3.5), (3.6), (3.7) and the fact that $F_1'(\eta_M)$, $F_2'(\eta_M) > 0$ into (3.8), we find that R.H.S. is positive whereas L.H.S. is non-positive. This proves the non-existence of η_M and implies that $F_1(\eta) = F_2(\eta)$ on $(0,\infty)$. Furthermore, the function $r'(\eta)$ which is positive on $(0,\eta_0)$ cannot attain maximum on $(0,\infty)$ but $r'(\infty) = F_1'(\infty) - F_2'(\infty) = 0$. This completes the proof of Theorem 3.2.

Now , the asymptotic behaviours, as $\eta \to \infty$, of the solutions of (2.10), (2.12) will be discussed based on the integrations of second order linear differential equations.

If $F(\eta)$ is the solution of (2.10), let us put

$$h(\eta) = 1 - F'(\eta) \tag{3.9}$$

Then $h(\eta)$ satisfies the differential equation

$$h'' + (\alpha_1\,\eta - F)h' - (\alpha_2 - F')h = 0 \tag{3.10}$$

Differentiating (3.10) gives\

$$h''' + (\alpha_1\,\eta - F)h'' + (\alpha_1 - \alpha_2 - 1 + F)h' = 0 \tag{3.11}$$

where $h' = -F''(\eta)$. In order to eliminate the middle term in (3.10), let us put

$$h = x\, exp\left[-\frac{1}{2}\int_0^\infty(\alpha_1\eta - F)d\eta\right] \tag{3.12}$$

to obtain

$$x'' - q(\eta)x = 0 \tag{3.12}$$

where

$$q(n) = \left(\frac{\alpha_1}{2} + \alpha_2\right) - \frac{3F'}{4} + \frac{1}{4}(\alpha_1\,\eta - F)^2$$
$$= \frac{1}{4}(\alpha_1\eta - F)^2[1 - \left[\frac{16F'}{(\alpha_1\eta - F)^2} + \frac{2\alpha_1 + 4\alpha_2}{(\alpha_1\eta - F)^2}\right] \tag{3.14}$$

Thus

$$q'(\eta) = -\frac{3}{2}F'' + \frac{1}{2}(\alpha_1\eta - F)(\alpha_1 - F')$$
$$q''(\eta) = \frac{9}{4}(\alpha_1\,\eta - F) + \frac{3}{2}(\alpha_2 - F')(1 - F') - \frac{3}{4}(\alpha_1 - F')^2$$

Since $0 < F' < 1$, $F'' > 0$ and $F' \sim 1$, $F \sim \eta$ as $\eta \to \infty$, there is a constants C' such that for large η

$$\frac{q'^2}{q^{\frac{5}{2}}} \le C'\left[\frac{F''^2}{\eta^5} + \frac{1}{\eta^3}\right], \frac{|q''|}{q^{\frac{3}{2}}} \le C'\left[\frac{F''}{\eta^2} + \frac{1}{\eta^3}\right]$$

In addition $\int^{\infty} F'' d\eta$ is absolutely convergent (since $F' \to 1$ so that $\eta \to \infty$) so that

$$\int^{\infty} \frac{q'^2}{q^{\frac{5}{2}}} d\eta < \infty \quad \text{and} \int^{\infty} \frac{|q''|}{q^{\frac{3}{2}}} d\eta < \infty \tag{3.15}$$

provided that

$$\int^{\infty} \frac{F''^2}{\eta^5} d\eta < \infty \tag{3.16}$$

It is easy to check the validity of (3.16), for an integration by parts (integrating F'' and differentiating $\frac{F''}{\eta^5}$) gives

$$\int \frac{F''^2}{\eta^5} d\eta = \frac{F'F''}{\eta^5} + \int \frac{F'}{\eta^5}[(\alpha_1 \eta - F)F'' + (\alpha_2 \eta - F')(1 - F')]d\eta$$

By (2.10), the last integral is absolutely convergent and $\lim inf\, F''(\eta) = 0$ as $\eta \to \infty$. Thus (3.16) holds. Consequently, (3.15) holds, and thus (3.13) has a principal solution $x(\eta)$ satisfying, as $\eta \to \infty$,

$$x \sim c_1 q^{-\frac{1}{4}}(\eta) \exp(-\int^{\eta} q^{\frac{1}{2}}(s) ds), \tag{3.17}$$

where $c_1 \neq 0$ is a constant, while linearly independent solution satisfy

$$x \sim c_1 q^{-\frac{1}{4}}(\eta) \exp(\int^{\eta} q^{\frac{1}{2}}(s) ds); \tag{3.18}$$

(cf. Exercise XI 9.6 Hartman [17], p. 382).

From the last part of (3.14) and $F \sim \eta$,

$$q^{\frac{1}{2}}(\eta) = \frac{1}{2}(\alpha_1 \eta - F) - \frac{3F'}{2(\alpha_1 \eta - F)} + \frac{\alpha_1 + 2\alpha_2}{2(\alpha_1 \eta - F)} + o\left(\frac{1}{\eta^3}\right)$$

$$q^{\frac{1}{4}}(\eta) \sim \left[\frac{1}{2}(\alpha_1 - 1)\eta\right]^{\frac{1}{2}}$$

hence

$$\int^{\eta} q^{\frac{1}{2}}(\eta) d\eta = \frac{1}{2}\int^{\eta}(\alpha_1 \eta - F)d\eta + \frac{3}{2}\log(\alpha_1 \eta - F) + (2\alpha_2 - \alpha_1)\int^{\eta} \frac{d\eta}{\alpha_1 \eta - F} + c^0 + o(1)$$

where c^0 is a constant.

Thus (3.17), (3.18) become

$$x \sim c_1 \eta \exp\left[-\frac{1}{2}\int^{\eta}(\alpha_1 \eta - F)d\eta\right],$$

$$x \sim c_1 \eta^{-2} \exp\left[\frac{1}{2}\int^{\eta}(\alpha_1 \eta - F)d\eta\right].$$

In view of (3.12), the equation (3.10) has a principal solution satisfying

$$h \sim c_1 \eta \exp\left[-\int^{\eta}\left\{\left(1 - \frac{A}{2}\right)\eta - F\right\}d\eta\right], \; c_1 \neq 0 \tag{3.19}$$

while the linearly independent solution satisfy

$$h \sim c_1 \eta^{-2}, \; c_1 \neq 0, \tag{3.20}$$

as $\eta \to \infty$.

By treating (3.11) as a second order equation for h' in the same way that (3.10) was handled, it is seen that (3.11) has the principal solutions satisfying, as $\eta \to \infty$,

$$h' = c_2 \eta^2 \exp\left[-\int^{\eta}\left\{\left(1 - \frac{A}{2}\right)\eta - F\right\}d\eta\right], \; c_1' \neq 0, \tag{3.21}$$

and the linearly independent solutions satisfy

$$h' = c_1' \eta^{-3}, \; c_1' \neq 0, \tag{3.22}$$

as $\eta \to \infty$.

If (3.9) satisfies (3.19), then since $F \sim \eta$, it follows that $\int^{\infty} h\eta \, d\eta < \infty$; thus

$$F = \eta + c_2 + o(1), \qquad \int^{\eta}\left\{\left(1 - \frac{A}{2}\right)\eta - F\right\}d\eta = -\frac{A\eta^2}{2} - c_2\eta + c_3 + o(1),$$

as $n \to \infty$.

Substituting this into (3.19), (3.21) gives

$$1 - F' \sim c_0 \eta \exp\left[-\frac{A\eta^2}{2} - c_2\eta\right], \; F'' \sim \eta(1 - F'), \tag{3.23}$$

as $\eta \to \infty$ where $c_0 > 0$, c_2 are the constants.

Similarly, from (3.20), (3.22) we obtain

$$1 - F' \sim c_0 \eta^{-2}, \; F'' \sim \eta^{-1}(1 - F'), \tag{3.24}$$

as $\eta \to \infty$.

Similarly based on the Exercise XI 9.6 of (Hartman [17], P.382), the equation (2.11) has the principal solution satisfying

$$\theta \sim c_3 \eta^{-\frac{3A}{2(1-A)}} \exp\left[-\frac{P_r(1-A)\eta^2}{2} - P_r c_2 \eta\right], \; c_3 \neq 0, \tag{3.25}$$

and the linearly independent solutions satisfy

$$\theta \sim c_3 \eta^{\frac{1+A}{2(1-A)}}, \; c_3 \neq 0, \tag{3.26}$$

as $\eta \to \infty$.

IV. Results

The study of the asymptotic behaviour of the solutions of similarity boundary layer equations, as the independent variable η tends to infinity; is of tremendous significance in fluid mechanics. If any particular solution is either zero, or infinitesimally small or is bounded for $\eta \to \infty$ it will show asymptotic character.

The results pertaining to the asymptotic behaviours of the principal and linearly independent solutions can be studied based on the criteria $1 - F' \to 0$, $F'' \to 0$ and $\theta \to 0$ as $\eta \to \infty$. We impose any one of these conditions, as the case may be, on the LHS of the solutions and see whether or not has the RHS same kind of behaviour as $\eta \to \infty$. If it is so, that particular solution will exhibit asymptotic character as $\eta \to \infty$, otherwise not.

Basin on the criteria that $1 - F' \to 0$ as $\to \infty$, we observe that the independent solutions (3.20), (3.22) will exhibit asymptotic behaviour as $\eta \to \infty$ whereas the principal solutions will not. As (3.20), (3.22) together lead to (3.24), the results in (3.24) will also show asymptotic behaviour as $\to \infty$. Likewise (3.19), (3.21) lead to (3.23). So the solution in (3.23) will not show asymptotic behaviour as $\eta \to \infty$.

To study the asymptotic behaviour of the solutions (3.25), (3.26) we take the criteria $\theta \to 0$ as $\eta \to \infty$ into account. From (3.25) it is obvious that it will behave asymptotically as $\eta \to \infty$. If $0 < A < 1$, but (3.26) will not. For $A > 1$ it will happen other way round. For $A > 1$, the independent solution (3.26) exhibits asymptotic behaviour as $\eta \to \infty$ whereas (3.25) do not.

V. Concluding Remarks

The asymptotic integration method to find out the solutions of non-linear boundary layer equations is the corner-stone of applied mathematics. This is the method to find the approximate solutions velocity and temperature profile for very large values of the independent variables. One of the other corner - stones of applied Mathematics is scientific computing and it is interesting to note that these two subjects have been grown together. However, this is not unexpected given their respective capabilities. By using computers, one is capable of solving problems that are non- linear, non-homogeneous and multi-dimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that, the computer solutions do not provide much insight into the physics of the problem, particularly for those which do not have access to the appropriate software or computer, and there is always a question as to whether or not is the computer solution correct. On the other hand, the asymptotic integration methods are also capable of finding the solutions of non-linear, non-homogeneous and multi dimensional problems. So the main objective behind the paper concerned is to provide reasonable, accurate expression for the solution for large values of η. By doing this one is able to derive an understanding of the physics of the problem.

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REFERENCES

[1] L. J. Crane, "Flow past a stretching plate," *J. Appl. Math. Phys.(ZAMP),* vol. 27, pp. 645-647, 1970.

[2] P. Carrangher, and L. J. Crane, "Heat transfer on a continuous stretching sheet," *J. Appl. Math. Mech. (ZAMM),* vol. 62, pp. 564-565, 1982.

[3] B. K. Dutta, P. Roy, and A. S. Gupta, "Temperature field in a flow over a stretching surface with uniform heat flux," *Int. Comm. Heat and Mass Transfer*, vol. 12, pp. 89-94, 1985.

[4] L. J. Grubka, and K. M. Bobaa, "Heat transfer characteristic of continuous stretching surface with variable temperature," *J. Heat Transfer,* vol. 107, pp. 248-250, 1985.

[5] E. M. A. Elbashbeshy, "Heat transfer over a stretching surface with a variable surface heat flux," *J. Phys. D: Appl. Phys.*, vol. 31, pp. 1951-1954, 1998.

[6] C. R. Lin, and C. K. Chen, "Exact solution of heat transfer from a starching surface with variable heat flux," *J. Heat Mass Transfer,* vol. 31, pp. 477-480, 1998.

[7] P. S. Gupta, and A. S. Gupta, " Heat and Mass transfer on a stretching sheet with suction and blowing," *Can. J. Chem. Eng.* , vol. 55, pp. 744-746, 1977.

[8] C. K. Chen, and M. I. Char, "Heat and mass transfer of continuous stretching surface with suction and blowing," *J. Math. Anal. Appl.* , vol. 135, pp. 568-580, 1988.

[9] E. Magyari, and B. Keller, "Heat and mass transfer in boundary layers on an exponentially stretching continuous surface," *J. Phys. D: Appl. Phys.*, vol. 32, pp. 577-585, 1999.

[10] E. Magyari, and B. Keller, "Exact solutions of the self similar boundary layer flows induced by permeable stretching surface," *Eur. J. Mech. B-Fluids,* vol. 19, pp. 109-122, 2000.

[11] S. J. Liao, and I. Pop, "Explicit analytic solution for similarity boundary layer equations," *Int. J. Heat and Mass Transfer*, vol. 33, pp. 75-85, 2004.

[12] T. Y. Na, and I. Pop, "Unsteady flow past a stretching sheet," *Mech. Res. Comm.*, vol. 23, pp. 413-422, 1996.

[13] C. Y. Wang, G. Due, M. Mikalvi, and C. C. Chang, "Impulsive stretching of a surface in a viscous fluid," *SIAM J. Appl. Math.*, vol. 57, pp. 1-14, 1997.

[14] R. Nazar, N. Amin, D. Filip, and I. Pop, "Unsteady boundary layer flow in the region of the stagnation point on the stretching sheet," *Int. J. Eng. Sci.*, vol. 42, pp. 1241-1253, 2004.

[15] E. M. A. Elbashbeshy, and M. A. A. Bazid, "Heat transfer over an unsteady stretching surface," *Heat and Mass transfer* , vol.41 1-4 2004.

[16] V. M. Falknar, and S. W. Skan, "Some approximate solutions of the boundary layer equations," *Phil. Mag.,* vol. 12 , 865-896 1931.

[17] D. R. Hartree, "On an equation occurring in Falknar- Skan's approximate treatment of the equations of the boundary layer," *Proc. Cambr. Phil. Soc.*, vol. 33 , pp. 233-239, 1937.

[18] H. Weyl, "On the differential equations of the simplest boundary layer problems," *Ann. Math.*, vol. 43, pp. 381-407, 1942.

[19] K. Stewartson, Further solutions of Falknar-Skan equation," *Proc. Cambr. Phil. Soc.*, vol. 50 , pp. 454-465, 1954.

[20] L. Rosenhead, *Laminar boundary layers* , Clarendon Press, Oxford, 1963 .

[21] S. P. Hasting, "Revered flow solutions of the Falknar-Skan equation," *SIAM J. Appl. Math.*, vol. 22 , pp. 330-334, 1972.

[22] W. C. Troy, "Non-monotonic solutions of the Falknar-Skan boundary layer equation," *Quart. Appl. Math.* , vol. 37, pp. 157-167, 1979.

[23] B. Gabutti, "On the equation of similar profiles," *J. Appl. Math. Phys. ZAMP)*, vol. 35 , pp. 265-281, 1984.

[24] B. B. Singh, and I. M. Chandarki, "Existence, uniqueness and asymptotic behaviours of MHD boundary layer flows of an incompressible fluid past a flat plate," *Int. J. Math. Sci. Engg. Appls.* , vol. 22, pp. 141-153 2008.

[25] P. Hartman, "On the asymptotic behaviours of solutions of a differential equations in boundary layer theory," *Z. Angew. Math. Mech. (ZAMM)* , vol. 44 , pp. 123-128, 1964.

[26] B. B. Singh, and A. K. Singh, "Asymptotic behaviours of solutions of MHD boundary layer flow on a moving continuous flat surface with heat flux," *J. Hunan Univ.*, vol. 16(1) , pp. 240-245, 1989.

[27] Singh, B. B. Asymptotic behaviours of hydromagnetic boundary layer flows past a semi-infinite flat plate," Riv. Math. Univ. Parma, 5(7), pp. 47-56, 2006.

[28] P. Hartman, *Ordinary differential equations*, 2nd ed., Birkhauser Boston, 1982 .